solution with the exact solution (within the framework of classical plate theory). It may be said in conclusion that the underlying assumption leading to Eq. (4) is adequate for support widths as large as 20% of the plate radius, insofar as the deflection and bending moments are concerned. This is borne out by Figs. 3–5. The approximate Ritz solution with its simple form may, therefore, be used with confidence. However, in those problems where the shear stress resultant in the support region is of interest the use of exact solution is imperative.

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Transient Response of a Cylindrical Shell Containing an Orthotropic Core

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A long, circular, cylindrical shell containing an annular, orthotropic, elastic core is subjected to an axisymmetric pressure pulse. The analysis considers the propagation and reflection of stress waves in the core, and the inner core boundary is taken as a rigid reflector or a free surface. Exact formulas in terms of elementary functions for the shell response are obtained for core materials which have specified relationships between the elastic constants. Graphical results from these special cases form closely spaced families of curves and results for an arbitrary orthotropic core can easily be obtained by interpolation.

Nomenclature

a = shell radius
b = inner radius of the core
c = radial dilatational wave speed in the core
c_{ij} = elastic constants of the core; see Eqs. (2a,b)
E = Young's modulus of the shell
E_r, E_θ, E_z = Young's moduli of the core
h = shell thickness
$H(T)$ = Heaviside unit function
I_α, K_α = modified Bessel functions of the first and second kind of order α
P = magnitude of step pressure pulse
q = radial shell displacement
r = radial coordinate
s = Laplace transform variable
t = time
T_D = duration of rectangular pressure pulse
u = radial displacement in the core
γ = shell density
ν = Poisson's ratio of the shell
$\nu_{\theta r}, \nu_{z\theta}, \nu_{zr}$ = Poisson's ratios of the core
ρ = core density
σ_r, σ_θ = radial and circumferential stress in the core

Dimensionless parameters

$c^2 = c_{11}/\rho$
$k = b/a$
$T = ct/a$
$W = q\omega^2/P$
$\alpha^2 = c_{22}/c_{11}$
$\beta^2 = \gamma c^2(1 - \nu^2)/E$
$\eta = r/a$
$\omega^2 = Eh/(1 - \nu^2)a^2$

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Introduction

COMPOSITE materials have recently improved the efficiency of structural elements for aerospace vehicles. Since many types of composite materials are possible, it is imperative to provide analytical methods which can guide the selection of efficient materials. The present analysis provides criteria for the selection or design of a core material which efficiently reduces the shell strains of a shell-core structure loaded by an axisymmetric pressure pulse. The core is specified as an orthotropic, elastic material. For this axisymmetric problem, the core has six independent material properties, three Young's moduli, and three Poisson's ratios; whereas an isotropic elastic core has only two independent material properties.

Some recent investigations associated with the propagation and reflection of elastic waves in transversely isotropic media have appeared in the literature and are closely related to the present analysis. Eason[1] considered the problem of stress waves emanating from spherical and cylindrical cavities in unbounded transversely isotropic media. Closed-form solutions were obtained for particular solids which had certain relations between the elastic constants. Bickford and Warren[2] extended the work of Ref. 1 and considered the propagation and reflection of elastic waves in isotropic and transversely isotropic, hollow spheres and cylinders subjected to time-dependent surface pressures.

The title problem was recently investigated for an isotropic core.[3] In this investigation, formulas in terms of elementary functions were developed for the shell response, and the results demonstrated that solutions which neglected inertial effects in the core could lead to significant errors. The present analysis generalizes the work of Ref. 3 by considering an orthotropic core. Shell response formulas are obtained for a long, circular, cylindrical shell containing an annular, orthotropic, elastic core subjected to an axisymmetric pressure pulse. The analysis considers the propagation and reflection of stress waves in the core, and the inner core boundary is taken as a rigid reflector or a free surface. Exact formulas in terms of elementary functions for the shell response are obtained for core materials which have specified relationships between the elastic constants. Graphical results from these special cases form closely spaced families of curves and results for an arbitrary orthotropic core can easily be obtained by interpolation.

Formulation

The shell-core structure shown in Fig. 1 is subjected to a circumferentially uniform, step-pressure pulse of magnitude P. For a shell of radius a, thickness h, Young's modulus E, Poisson's ratio ν, and density γ, the radial shell motion is governed by the equation

$$\beta^2 \frac{\partial^2 W}{\partial T^2} + W = H(T) + \frac{\sigma_r(a,T)}{P} \quad (1a)$$

$$W = q\omega^2/P; \quad T = ct/a; \quad \beta^2 = \gamma c^2(1 - \nu^2)/E$$
$$\omega^2 = Eh/(1 - \nu^2)a^2 \quad (1b)$$

where q is the radial shell displacement, $\sigma_r(a,T)$ is the stress in the core at the shell-core interface, $H(T)$ is the Heaviside step function, t is time, and c is the velocity of a radially propagating dilatational wave in the core.

The problem is one of axisymmetric plane-strain, and the stress-displacement relations for the orthotropic core are

$$\sigma_r = c_{11}\partial u/\partial r + c_{12}(u/r) \quad (2a)$$

$$\sigma_\theta = c_{12}\partial u/\partial r + c_{22}(u/r) \quad (2b)$$

where r is the radial coordinate; σ_r, σ_θ are the radial and circumferential components of stress, respectively; u is the radial displacement; and c_{ij} are the elastic constants.§ The radial displacement is governed by the wave equation

$$\partial^2 u/\partial\eta^2 + (1/\eta)(\partial u/\partial\eta) - \alpha^2 u/\eta^2 = \partial^2 u/\partial T^2 \quad (3a)$$

$$\eta = r/a; \quad \alpha^2 = c_{22}/c_{11}; \quad c^2 = c_{11}/\rho \quad (3b)$$

where ρ is the density of the core. A Laplace transform is taken over the variable T, and Eq. (3a) has the solution

$$\bar{u}(\eta,s) = A(s)I_\alpha(\eta s) + B(s)K_\alpha(\eta s) \quad (4)$$

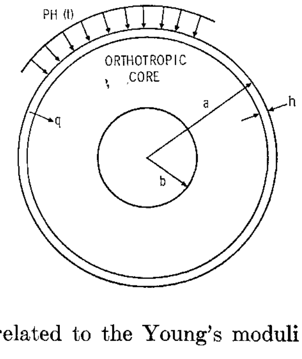

Fig. 1 Geometry of the problem.

§ The elastic constants c_{ij} are related to the Young's moduli and Poisson's ratios in Ref. 4.

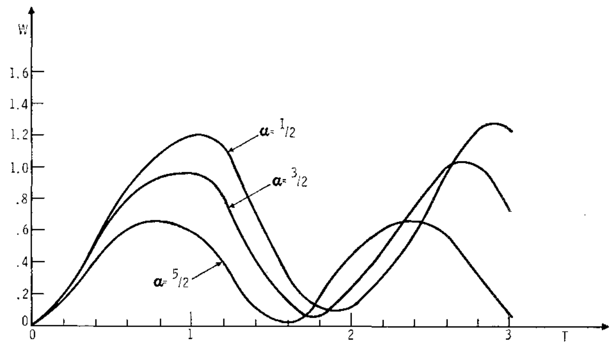

Fig. 2a Shell response for a rigid reflector, $E_r/E = 0.01$.

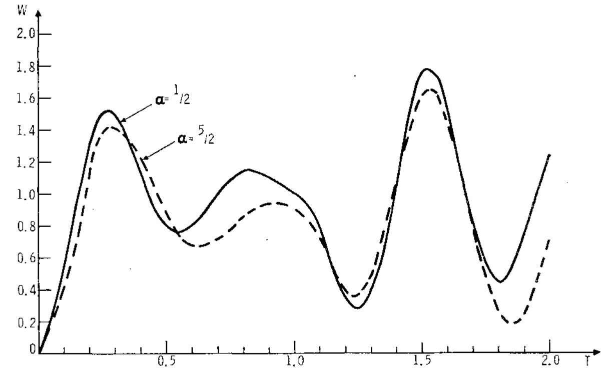

Fig. 2b Shell response for a rigid reflector, $E_r/E = 0.001$.

where $\bar{u}(\eta,s)$ is the transform of $u(\eta,T)$, s is the transform variable, and I_α, K_α are modified Bessel functions of the first and second kind of order α. The transformed radial stress component is

$$\bar{\sigma}_r(\eta,s) = \frac{A(s)}{a\eta}[(c_{11}\alpha + c_{12})I_\alpha(\eta s) + c_{11}\eta s I_{\alpha+1}(\eta s)] +$$
$$\frac{B(s)}{a\eta}[(c_{11}\alpha + c_{12})K_\alpha(\eta s) - c_{11}\eta s K_{\alpha+1}(\eta s)] \quad (5a)$$

and the transform of Eq. (1a), the shell equation of motion, is

$$(\beta^2 s^2 + 1)\bar{W} = 1/s + \bar{\sigma}_r(1,s)/P \quad (5b)$$

The functions $A(s)$ and $B(s)$ are determined from the boundary conditions at the shell-core interface and at the core inner boundary. The shell is taken as bonded to the core, and continuity of displacement is maintained at the interface. Thus, the transformed boundary condition at $r = a$ or $\eta = 1$ is

$$\bar{W}(s) + (\omega^2/P)\bar{u}(1,s) = 0 \quad (6)$$

For this analysis, the boundary conditions at the core inner radius are specified as a rigid reflector or a free surface. Exact formulas in terms of elementary functions for the shell response are obtained for particular values of the parameter α. Graphical results from these special cases form closely spaced families of curves and results for arbitrary orthotropic cores can be obtained by interpolation.

Shell Response for a Rigid Reflector at the Inner Core Boundary

The ratio of the inner to outer core radius $k = b/a$ is defined. Then the transformed boundary condition at $\eta = k$ for a rigid reflector is

$$\bar{u}(k,s) = 0 \quad (7)$$

The functions $A(s)$ and $B(s)$ can now be evaluated from Eqs. (6) and (7) substituted into the expression for the transformed radial stress given by Eq. (5a). The value of σ_r at $\eta = 1$ is then substituted into Eq. (5b), the transformed shell equation of motion, and the formal solution for the shell response is given by

$$\bar{W} = \frac{K_\alpha(ks)I_\alpha(s) - K_\alpha(s)I_\alpha(ks)}{s\{[\beta^2 s^2 + 1 - (\alpha c_{11}/\alpha c_{11}/a\omega^2 - c_{12}/a\omega^2)][K_\alpha(ks)I_\alpha(s) - K_\alpha(s)I_\alpha(ks)] + (c_{11}s/a\omega^2) \times [K_\alpha(ks)I_{\alpha-1}(s) + K_{\alpha-1}(s)I_\alpha(ks)]\}} \quad (8)$$

Equation (8) contains the modified Bessel functions I_α, K_α; and these functions seem to eliminate the possibility of an exact inversion for the general case. However, when $\alpha = \pm n/2$ where n is a positive integer, the modified Bessel functions are expressed in terms of elementary functions which permits an exact inversion. In this section Eq. (8) is inverted for $\alpha = \frac{1}{2}, \frac{3}{2}$, and $\frac{5}{2}$.

Special Case Where $\alpha = 1/2$

From Ref. 5,

$$I_{1/2}(s) = (2/\pi s)^{1/2} \sinh s \quad (9a)$$

$$I_{-1/2}(s) = (2/\pi s)^{1/2} \cosh s \quad (9b)$$

$$K_{1/2}(s) = K_{-1/2}(s) = (\pi/2s)^{1/2} e^{-s} \quad (9c)$$

and Eq. (8) reduces to

$$\bar{W} = \frac{1}{\beta^2 s F(s)} \left\{ \frac{1 - e^{-2s(1-k)}}{1 - [G(s)/F(s)]e^{-2s(1-k)}} \right\} \quad (10a)$$

in which

$$F(s) = s^2 + (c_{11}/a\omega^2\beta^2)s + (1 - c_{11}/2a\omega^2 + c_{12}/a\omega^2)/\beta^2 \quad (10b)$$

and

$$G(s) = s^2 - (c_{11}/a\omega^2\beta^2)s + (1 - c_{11}/2a\omega^2 + c_{12}/a\omega^2)/\beta^2 \quad (10c)$$

Eq. (10a) is expanded for positive s, (e.g., see Ref. 6). Thus

$$\bar{W} = \frac{1}{\beta^2 s F(s)} - \frac{2c_{11}e^{-2s(1-k)}}{a\omega^2\beta^4[F(s)]^2} - \frac{2c_{11}G(s)e^{-4s(1-k)}}{a\omega^2\beta^4[F(s)]^3} + \ldots \quad (11)$$

Equation (11) is inverted by standard techniques and

$$W = W_1 + W_2 H(T_1) + W_3 H(T_2) + \ldots \quad (12a)$$

in which

$$T_1 = T - 2(1-k), \quad T_2 = T - 4(1-k) \quad (12b)$$

$$W_1 = \frac{2}{2 - d(4j - 1)} + \frac{e^{lT}}{fl} - \frac{e^{mT}}{fm} \quad (12c)$$

$$W_2 = \frac{-2d}{f^2}\left[e^{lT_1}\left(T_1 - \frac{2\beta^2}{f}\right) + e^{mT_1}\left(T_1 + \frac{2\beta^2}{f}\right)\right] \quad (12d)$$

$$W_3 = \frac{2d^2}{f^3}\left\{le^{lT_2}\left[\left(T_2 + \frac{1}{2l} - \frac{\beta^2}{d} - \frac{3\beta^2}{f}\right)^2 - \frac{1}{4l^2} - \frac{\beta^4}{d^2} + \frac{3\beta^4}{f^2}\right] - me^{mT_2}\left[\left(T_2 + \frac{1}{2m} - \frac{\beta^2}{d} + \frac{3\beta^2}{f}\right)^2 - \frac{1}{4m^2} - \frac{\beta^4}{d^2} + \frac{3\beta^4}{f^2}\right]\right\} \quad (12e)$$

where l, m are the roots of $F(s) = 0$ with $F(s)$ given by Eq. (10b) and

$$d = c_{11}/a\omega^2, \quad j = (1 - c_{12}/c_{11})/2 \quad (12f)$$

$$f = [d^2 - 4\beta^2(1 - 2dj + d/2)]^{1/2} \quad (12g)$$

For $l = m$ the form of solution would change and it is assumed in the analysis that $l \neq m$.

The time it takes the dilatational wave front traveling in the core to strike the rigid reflector, propagate back, and strike the shell is $T = 2(1-k)$. Then Eq. (12a) is valid for $0 < T < 6(1-k)$, which is sufficient time to adequately describe the shell response.

Special Case Where $\alpha = 3/2$

The response formula for $\alpha = \frac{3}{2}$ can be found by using the procedure outlined in the case for $\alpha = \frac{1}{2}$. Thus, for $\alpha = \frac{3}{2}$

$$\bar{W} = \frac{s-1}{\beta^2 s F(s)} \times \left\{ \frac{1 - [(ks-1)(s+1)/(ks+1)(s-1)]e^{-2s(1-k)}}{1 - [(ks-1)G(s)/(ks+1)F(s)]e^{-2s(1-k)}} \right\} \quad (13a)$$

where

$$F(s) = s^3 - ds^2 + js - j \quad (13b)$$

$$G(s) = s^3 + ds^2 + js + j \quad (13c)$$

$$d = 1 - c_{11}/a\omega^2\beta^2, \quad j = (1 - 3c_{11}/2a\omega^2 + c_{12}/a\omega^2)/\beta^2 \quad (13d)$$

The shell response is given by Eqs. (12a,b) with

$$\beta^2 W_1 = \frac{1}{s_1 s_2 s_3} + \sum_{i=1}^{3} \frac{(s_i - 1)e^{s_i T}}{s_i(s_i - s_m)(s_i - s_n)} \quad (14a)$$

$$\frac{a\omega^2\beta^4}{4c_{11}} W_2 = \frac{k^3 e^{-T_1/k}}{(1+ks_1)^2(1+ks_2)^2(1+ks_3)^2} - \sum_{i=1}^{3} \frac{s_i^2(ks_i - 1)e^{s_i T_1}}{(ks_i + 1)(s_i - s_m)^2(s_i - s_n)^2} \times \left[\frac{1}{s_i} + \frac{k}{k^2 s_i^2 - 1} - \frac{1}{s_i - s_m} - \frac{1}{s_i - s_n} + \frac{T_1}{2}\right] \quad (14b)$$

$$\frac{a\omega^2\beta^4}{c_{11}} W_3 = \frac{24k^3(1 - dk + jk^2 - jk^3)e^{-T_2/k}}{(1+ks_1)^3(1+ks_2)^3(1+ks_3)^3} \times \left[1 + \frac{3 - 2dk + jk^2}{3(1 - dk + jk^2 - jk^3)} - \frac{1}{1+ks_1} - \frac{1}{1+ks_2} - \frac{1}{1+ks_3} - \frac{T_2}{3k}\right] - \sum_{i=1}^{3} \frac{s_i^2(ks_i - 1)^2(s_i^3 + ds_i^2 + js_i + j)e^{s_i T_2}}{(ks_i + 1)^2(s_i - s_m)^3(s_i - s_n)^3} \times \left[\left(\frac{2}{s_i} + \frac{4k}{k^2 s_i^2 - 1} + \frac{3s_i^2 + 2ds_i + j}{s_i^3 + ds_i^2 + js_i + j} - \frac{3}{s_i - s_m} - \frac{3}{s_i - s_n} + T_2\right)^2 - \frac{2}{s_i^2} - \frac{8k^3 s_i}{(k^2 s_i^2 - 1)^2} + \frac{6s_i + 2d}{s_i^3 + ds_i^2 + js_i + j} - \left(\frac{3s_i^2 + 2ds_i + j}{s_i^3 + ds_i^2 + js_i + j}\right)^2 + \frac{3}{(s_i - s_m)^2} + \frac{3}{(s_i - s_n)^2}\right] \quad (14c)$$

where s_i are the roots of $F(s) = 0$ with $F(s)$ given by Eq. (13b); m and n are integers that vary from 1 to 3; $i \neq m$, $i \neq n$, and $m \neq n$ in any one term. It is assumed in the analysis that none of the roots of $F(s)$ are equal.

Special Case Where $\alpha = 5/2$

The transformed solution for $\alpha = \frac{5}{2}$ is given by

$$\bar{W} = \frac{(s^2 - 3s + 3)}{\beta^2 sF(s)} \times \left\{\frac{1 - [(s^2 + 3s + 3)(k^2s^2 - 3ks + 3)/(s^2 - 3s + 3)(k^2s^2 + 3ks + 3)]e^{-2s(1-k)}}{1 - [(k^2s^2 - 3ks + 3)G(s)/(k^2s^2 + 3ks + 3)F(s)]e^{-2s(1-k)}}\right\} \quad (15a)$$

where

$$F(s) = s^4 - ds^3 + fs^2 - js + j \quad (15b)$$

$$G(s) = s^4 + ds^3 + fs^2 + js + j \quad (15c)$$

$$d = 3 - c_{11}/a\omega^2\beta^2; \quad f = 3 + (1 - 7c_{11}/2a\omega^2 + c_{12}/a\omega^2)/\beta^2$$

$$j = 3(1 - 5c_{11}/2a\omega^2 + c_{12}/a\omega^2)/\beta^2 \quad (15d)$$

The shell response is given by Eqs. (12a,b) with

$$\beta^2 W_1 = \frac{3}{s_1s_2s_3s_4} + \sum_{i=1}^{4} \frac{(s_i^2 - 3s_i + 3)e^{s_iT}}{s_i(s_i - s_m)(s_i - s_n)(s_i - s_p)} \quad (16a)$$

$$\frac{k^2a\omega^2\beta^4}{2c_{11}} W_2 = - \sum_{i=1}^{2} \frac{x_i^4(k^2x_i^2 - 3kx_i + 3)e^{x_iT_1}}{(x_i - x_q)(x_i - s_1)^2(x_i - s_2)^2(x_i - s_3)^2(x_i - s_4)^2} - \sum_{i=1}^{4} \frac{s_i^4(k^2s_i^2 - 3ks_i + 3)e^{s_iT_1}}{(s_i - s_m)^2(s_i - s_n)^2(s_i - s_p)^2(s_i - x_1)(s_i - x_2)} \times \left[\frac{6k^2s_i^2 - 15ks_i + 12}{s_i(k^2s_i^2 - 3ks_i + 3)} - \frac{2}{s_i - s_m} - \frac{2}{s_i - s_n} - \frac{2}{s_i - s_p} - \frac{1}{s_i - x_1} - \frac{1}{s_i - x_2} + T_1\right] \quad (16b)$$

$$\frac{a\omega^2k^4\beta^4}{2c_{11}} W_3 = - \sum_{i=1}^{2} \frac{x_i^4(k^2x_i^2 - 3kx_i + 3)^2(x_i^4 + dx_i^3 + fx_i^2 + jx_i + j)e^{x_iT_2}}{(x_i - x_q)^2(x_i - s_1)^3(x_i - s_2)^3(x_i - s_3)^3(x_i - s_4)^3} \times \left[\frac{4}{x_i} + \frac{2k(2kx_i - 3)}{k^2x_i^2 - 3kx_i + 3} + \frac{4x_i^3 + 3dx_i^2 + 2fx_i + j}{x_i^4 + dx_i^3 + fx_i^2 + jx_i + j} - \frac{2}{x_i - x_q} - \frac{3}{x_i - s_1} - \frac{3}{x_i - s_2} - \frac{3}{x_i - s_3} - \frac{3}{x_i - s_4} + T_2\right] - \sum_{i=1}^{4} \frac{s_i^4(k^2s_i^2 - 3ks_i + 3)^2(s_i^4 + ds_i^3 + fs_i^2 + js_i + j)e^{s_iT_2}}{2(s_i - x_1)^2(s_i - x_2)^2(s_i - s_m)^3(s_i - s_n)^3(s_i - s_p)^3} \times \left\{\left[\frac{4}{s_i} + \frac{2k(2ks_i - 3)}{k^2s_i^2 - 3ks_i + 3} + \frac{4s_i^3 + 3ds_i^2 + 2fs_i + j}{s_i^4 + ds_i^3 + fs_i^2 + js_i + j} - \frac{2}{s_i - x_1} - \frac{2}{s_i - x_2} - \frac{3}{s_i - s_m} - \frac{3}{s_i - s_n} - \frac{3}{s_i - s_p} + T_2\right]^2 - \frac{4}{s_i^2} - \frac{2k^2(2k^2s_i^2 - 6ks_i + 3)}{(k^2s_i^2 - 3ks_i + 3)^2} + \frac{12s_i^2 + 6ds_i + 2f}{s_i^4 + ds_i^3 + fs_i^2 + js_i + j} - \left(\frac{4s_i^3 + 3ds_i^2 + 2fs_i + j}{s_i^4 + ds_i^3 + fs_i^2 + js_i + j}\right)^2 + \frac{2}{(s_i - x_1)^2} + \frac{2}{(s_i - x_2)^2} + \frac{3}{(s_i - s_m)^2} + \frac{3}{(s_i - s_n)^2} + \frac{3}{(s_i - s_p)^2}\right\} \quad (16c)$$

where s_i are the roots of $F(s) = 0$ with $F(s)$ given by Eq. (15b); x_i are the roots of

$$s^2 + 3s/k + 3/k^2 = 0 \quad (16d)$$

m, n, p are integers that vary from 1 to 4; q is an integer equal to 1 or 2; $i \neq m$, $i \neq n$, $i \neq p$, $m \neq n$, $m \neq p$, in each term; $q \neq i$ in each term. As in the previous cases, it is assumed that the roots of $F(s) = 0$ and Eq. (16d) are not equal. If these roots were equal the form of solution would change.

Shell Response for a Free Surface at the Inner Core Boundary

The transformed boundary condition at $\eta = k$ for a free surface is

$$\bar{\sigma}_r(k,s) = 0 \quad (17)$$

and the transformed shell response is given by

$$s\bar{W} = \frac{I_\alpha(s)H_1(ks) - K_\alpha(s)H_2(ks)}{\left[\beta^2s^2 + 1 + \dfrac{c_{12} - \alpha c_{11}}{a\omega^2}\right]\left[I_\alpha(s)H_1(ks) - K_\alpha(s)H_2(ks) + \dfrac{c_{11}s}{a\omega^2}\left[I_{\alpha-1}(s)H_1(ks) + K_{\alpha-1}(s)H_2(ks)\right]\right]} \quad (18a)$$

where

$$H_1(ks) = (c_{12} - \alpha c_{11})K_\alpha(ks) - c_{11}ksK_{\alpha-1}(ks) \quad (18b)$$

$$H_2(ks) = (c_{12} - \alpha c_{11})I_\alpha(ks) + c_{11}ksI_{\alpha-1}(ks) \quad (18c)$$

In this section Eq. (18a) is inverted for $\alpha = \frac{1}{2}$ and $\frac{3}{2}$.

Special Case Where $\alpha = 1/2$

For $\alpha = \frac{1}{2}$, Eq. (18a) reduces to

$$\bar{W} = \frac{1}{\beta^2sF(s)}\left\{\frac{1 + [(s - g)/(s + g)]e^{-2s(1-k)}}{1 + [(s - g)G(s)/(s + g)F(s)]e^{-2s(1-k)}}\right\} \quad (19a)$$

where $F(s)$ and $G(s)$ are given by Eqs. (10b,c) and

$$g = (\tfrac{1}{2} - c_{12}/c_{11})/k \quad (19b)$$

The shell response is given by Eqs. (12a,b,c) and

$$W_2 = \frac{-4dge^{-gT_1}}{\beta^4(g + l)^2(g + m)^2} + \frac{2de^{lT_1}}{f^2(l + g)} \times \left[\frac{2g}{l + g} - \frac{2(l - g)}{l - m} + (l - g)T_1\right] + \frac{2de^{mT_1}}{f^2(m + g)} \times \left[\frac{2g}{m + g} + \frac{2(m - g)}{l - m} + (m - g)T_1\right] \quad (20a)$$

$$W_3 = \frac{-8dg^2(l - g)(m - g)e^{-gT_2}}{\beta^4(l + g)^3(m + g)^3}\left[T_2 - \frac{1}{g} + \frac{2(2l - g)}{l^2 - g^2} + \frac{2(2m - g)}{m^2 - g^2}\right] + \frac{2d^2l(l - g)^2e^{lT_2}}{f^3(l + g)^2}\left\{\frac{2}{(l + g)^2} + \frac{3}{(l - m)^2} - \frac{2}{(l - g)^2} - \frac{1}{(l + m)^2} - \frac{1}{4l^2} + \left[T_2 + \frac{4g}{l^2 - g^2} - \frac{2(2m + l)}{l^2 - m^2} + \frac{1}{2l}\right]^2\right\} - \frac{2d^2m(m - g)^2e^{mT_2}}{f^3(m + g)^2} \times \left\{\frac{2}{(m + g)^2} + \frac{3}{(m - l)^2} - \frac{2}{(m - g)^2} - \frac{1}{(m + l)^2} - \frac{1}{4m^2} + \left[T_2 + \frac{4g}{m^2 - g^2} - \frac{2(2l + m)}{m^2 - l^2} + \frac{1}{2m}\right]^2\right\} \quad (20b)$$

where l and m are the roots of $F(s) = 0$ with $F(s)$ given by Eq. (10b); it is assumed $l \neq m$; d, j, and f are defined by Eqs. (12f,e).

The shell response for W_1 is the same for all inner core boundary conditions because the signal transmitted back to the shell from the inner core boundary does not arrive at the shell-core interface until $T = 2(1 - k)$.

Special Case Where $\alpha = 3/2$

The transformed shell response for $\alpha = \frac{3}{2}$ is given by

$$\bar{W} = \frac{(s-1)}{\beta^2 s F(s)} \times \left\{\frac{1 + [(s+1)(s^2 - gs + f)/(s-1)(s^2 + gs + f)]e^{-2s(1-k)}}{1 + [(s^2 - gs + f)G(s)/(s^2 + gs + f)F(s)]e^{-2s(1-k)}}\right\} \qquad (21a)$$

where $F(s)$, $G(s)$, d, j are defined by Eqs. (13b,c,d) and

$$f = (\tfrac{3}{2} - c_{12}/c_{11})/k^2; \quad g = (\tfrac{3}{2} - c_{12}/c_{11})/k \qquad (21b)$$

The shell response is given by Eqs. (12a,b), Eq. (14a), and

$$\frac{a\omega^2\beta^4}{2c_{11}} W_2 = \sum_{i=1}^{2} \frac{x_i^2(x_i^2 - gx_i + f)e^{x_iT_1}}{(x_i - x_q)(x_i - s_1)^2(x_i - s_2)^2(x_i - s_3)^2} + \sum_{i=1}^{3} \frac{s_i^2(s_i^2 - gs_i + f)e^{s_iT_1}}{(s_i - x_1)(s_i - x_2)(s_i - s_m)^2(s_i - s_n)^2} \times \left[\frac{2}{s_i} + \frac{2s_i - g}{s_i^2 - gs_i + f} - \frac{1}{s_i - x_1} - \frac{1}{s_i - x_2} - \frac{2}{s_i - s_m} - \frac{2}{s_i - s_n} + T_1\right] \qquad (22a)$$

$$\frac{a\omega^2\beta^4}{2c_{11}} W_3 = - \sum_{i=1}^{2} \frac{x_i^2(x_i^2 - gx_i + f)^2(x_i^3 + dx_i^2 + jx_i + j)e^{x_iT_2}}{(x_i - x_q)^2(x_i - s_1)^3(x_i - s_2)^3(x_i - s_3)^3} \times \left[\frac{2}{x_i} + \frac{2(2x_i - g)}{x_i - gx_i + f} + \frac{3x_i^2 + 2dx_i + j}{x_i^3 + dx_i^2 + jx_i + j} - \frac{2}{x_i - x_q} - \frac{3}{x_i - s_1} - \frac{3}{x_i - s_2} - \frac{3}{x_i - s_3} + T_2\right] - \sum_{i=1}^{3} \frac{s_i^2(s_i^2 - gs_i + f)^2(s_i^3 + ds_i^2 + js_i + j)e^{s_iT}}{2(s_i - x_1)^2(s_i - x_2)^2(s_i - s_m)^3(s_i - s_n)^3} \times \left\{\left[\frac{2}{s_i} + \frac{2(2s_i - g)}{s_i^2 - gs_i + f} + \frac{3s_i^2 + 2ds_i + j}{s_i^3 + ds_i^2 + js_i + j} - \frac{2}{s_i - x_1} - \frac{2}{s_i - x_2} - \frac{3}{s_i - s_m} - \frac{3}{s_i - s_n} + T_2\right]^2 - \frac{2}{s_i^2} + \frac{4}{s_i^2 - gs_i + f} - \frac{2(2s_i - g)^2}{(s_i^2 - gs_i + f)^2} + \frac{6s_i + 2d}{s_i^3 + ds_i^2 + js_i + j} - \left(\frac{3s_i^2 + 2ds_i + j}{s_i^3 + ds_i^2 + js_i + j}\right)^2 + \frac{2}{(s_i - x_1)^2} + \frac{2}{(s_i - x_2)^2} + \frac{3}{(s_i - s_m)^2} + \frac{3}{(s_i - s_n)^2}\right\} \qquad (22b)$$

where x_i are the roots of

$$s^2 + gs + f = 0 \qquad (22c)$$

s_i are the roots of $F(s) = 0$ with $F(s)$ given by Eq. (13b); m and n are integers that vary from 1 to 3; $i \neq m$, $i \neq n$, and $m \neq n$ in any one term; q is an integer that has values 1 or 2; $q \neq i$ in any one term. It is assumed in the analysis that $F(s)$ and Eq. (22c) have unequal roots.

Discussion and Numerical Results

Shell response formulas have been derived for core materials which have specified relationships between the elastic constants. The formal solution for an arbitrary orthotropic core is given by Eq. (8) for the case of a rigid reflector at the inner core boundary. This solution contains modified Bessel functions of the first and second kind, and these functions seem to eliminate an exact inversion for the general case. However, when $\alpha = \pm n/2$ where n is an integer and $\alpha^2 = c_{22}/c_{11}$ the modified Bessel functions are expressed in terms of elementary functions. For an isotropic core, $c_{22} = c_{11}$ and $\alpha = 1$; then $\alpha = \frac{1}{2}$ and $\alpha = \frac{3}{2}$ bracket the isotropic case. In this analysis, formulas are presented for $\alpha = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$ when the inner core boundary is a rigid reflector and for $\alpha = \frac{1}{2}, \frac{3}{2}$ when the inner core boundary is a free surface. Response formulas for other values of n where $\alpha = \pm n/2$ can be calculated with the procedure outlined in this analysis; however, the algebraic computations required to derive the shell response formulas increase as n increases.

It is convenient in the analysis to use the stress-strain relationships for the core in the form of Eqs. (2a,b). However, since material property data are given in terms of Young's moduli E_r, E_θ, E_z and Poisson's ratios $\nu_{z\theta}$, ν_{zr}, $\nu_{\theta r}$, the elastic constants c_{ij} must be related to Young's moduli and Poisson's ratios. These relationships are given in Ref. 4. Also, there are certain constraints among the elastic constants which must be satisfied; these constraints are recorded in Ref. 4.

Response data for all the numerical examples are based on the following common parameters: $a/h = 40$; $k = 0.5$; $\rho/\gamma = 0.125$; $\nu = 0.333$; $\nu_{z\theta} = \nu_{\theta r} = \nu_{zr} = 0.125$; and

$$\frac{E_\theta}{E_r} = \frac{E_z}{E_r} = \begin{cases} 0.25 & (\alpha = \frac{1}{2}) \\ 2.30 & (\alpha = \frac{3}{2}) \\ 6.90 & (\alpha = \frac{5}{2}) \end{cases}$$

The other required parameters, E_r/E and the inner core boundary condition, are specified on each figure. The notation for the core material properties is the same as that

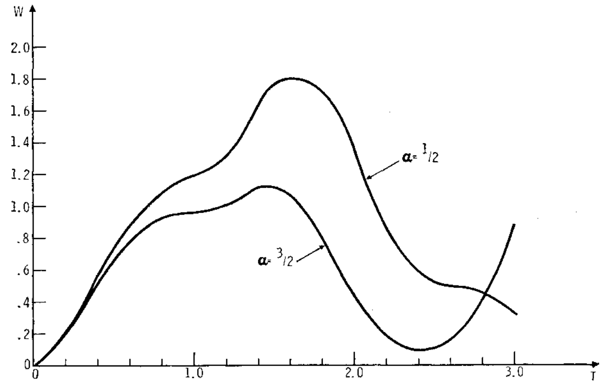

Fig. 3a Shell response for a free surface, $E_r/E = 0.01$.

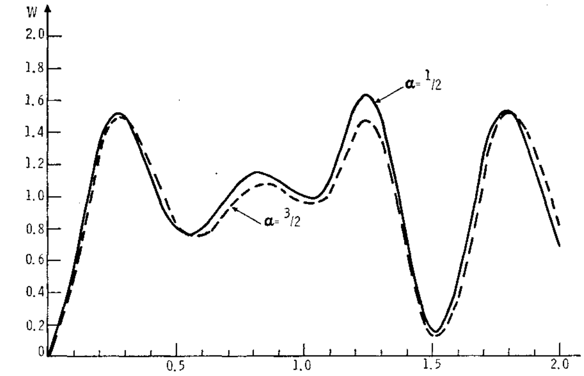

Fig. 3b Shell response for a free surface, $E_r/E = 0.001$.

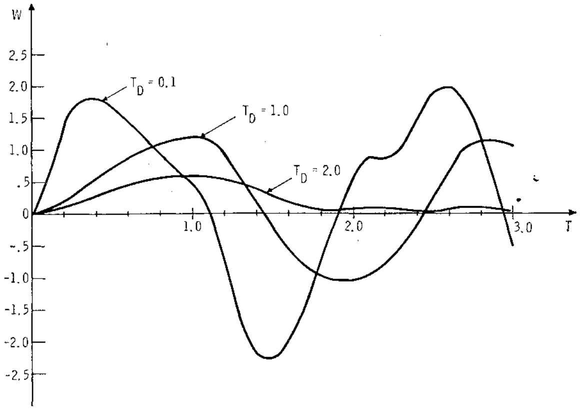

Fig. 4a Shell response for a rigid reflector, $E_r/E = 0.01$, $\alpha = 1/2$.

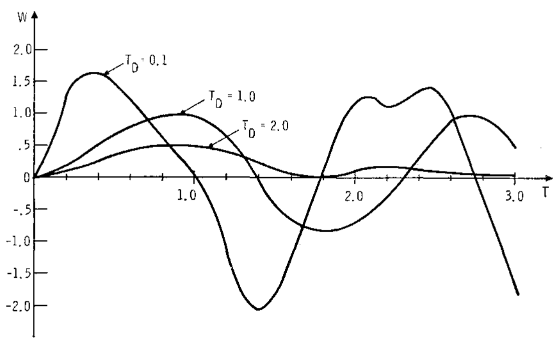

Fig. 4b Shell response for a rigid reflector, $E_r/E = 0.01$, $\alpha = 3/2$.

used in Refs. 4 and 7. Shell response curves for a shell-core structure with a rigid reflector and a free surface as the inner core boundaries are presented in Figs. 2a,b and Figs. 3a,b, respectively. The shell-core structures are subjected to a circumferentially uniform, step-pressure pulse of magnitude P. The ordinate of the response curves is $W = q\omega^2/P$, where P/ω^2 is the shell displacement which would occur without a core if the load P were applied statically. Then if there were no core, the maximum dynamic response would be $W = 2$. Since the problem is axisymmetric, the corresponding hoop strain and stress for the shell are $\epsilon = q/a$ and $\sigma = E\epsilon/(1 - \nu^2)$.

Shell response curves for a shell-core structure with a rigid reflector as the inner core boundary are presented in Figs. 4a,b. For these response curves, the shell-core structures are subjected to circumferentially uniform, rectangular pulses of duration T_D and amplitude P/T_D. These curves demonstrate the effect of pulse duration on the shell response for three pulses with a constant impulse.

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